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The minimum cardinality of maximal systems of rectangular islands

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Abstract

For given positive integers m and n, and $R = \{(x, y) : 0 \leq x \leq m \text{ and } 0 \leq y \leq n\}$, a set H of rectangles that are all subsets of R and the vertices of which have integer coordinates is called a system of rectangular islands if for every pair of rectangles in H one of them contains the other or they do not overlap at all. Let I_R denote the ordered set of systems of rectangular islands on R, and let $\max(I_R)$ denote the maximal elements of I_R. For $f(m, n) = \max\{|H| : H \in \max(I_R)\}$, G. Czédli [G. Czédli, The number of rectangular islands by means of distributive lattices, European J. Combin. 30 (1) (2009) 208–215)] proved $f(m, n) = \lfloor (mn + m + n - 1)/2 \rfloor$. For $g(m, n) = \min\{|H| : H \in \max(I_R)\}$, we prove $g(m, n) = m + n - 1$. We also show that for any integer h in the interval $[g(m, n), f(m, n)]$, there exists an $H \in \max(I_R)$ such that $|H| = h$.

1. Introduction

The notion of *systems of rectangular islands* was introduced by G. Czédli [1]. For integers $m > 0$, and $n > 0$, consider the rectangle $R = \{(x, y) : 0 \leq x \leq m \text{ and } 0 \leq y \leq n\}$ in the Cartesian Plane. A set of rectangles that are all subsets of R and the vertices of which have integer coordinates form a system of rectangular islands H if for every pair of rectangles $R_1, R_2 \in H$ either $R_1 \subseteq R_2$, or $R_2 \subseteq R_1$, or $R_1 \cap R_2 = \emptyset$.

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The main theorem in [1] states that the maximum cardinality of a system of rectangular islands is given by

$$f(m, n) = \left\lfloor \frac{mn + m + n - 1}{2} \right\rfloor.$$

Systems of rectangular islands on a given $m \times n$ rectangle R form a partially ordered set I_R with respect to set inclusion. If $\max(I_R)$ denotes the subset of maximal elements of I_R, then Czédli's result can be interpreted as finding a formula for

$$f(m, n) = \max\{|H| : H \in \max(I_R)\}.$$

The purpose of this paper is to determine

$$g(m, n) = \min\{|H| : H \in \max(I_R)\}.$$

Further, we provide a description of maximal systems of rectangular islands that achieve the minimum cardinality $g(m, n)$. We also show that any integer between $g(m, n)$ and $f(m, n)$ occurs as the cardinality of some maximal system of rectangular islands.

2. The minimum

Proposition 1. $g(m, n) = m + n - 1$.

Proof. First note that $g(m, n) \leq m + n - 1$ follows from the fact that there is a sequence of $m + n - 1$ increasing rectangles, each included in the next, which form a maximal system of rectangular islands. Hence, it is enough to show that $g(m, n) \geq m + n - 1$, and we will proceed by induction on $m + n$.

Let $\max(H)$ denote the set of maximal rectangles (with respect to set inclusion) in $H - \{R\}$, and for a given rectangle $Q \in H$, define $H|_Q = \{S \in H : S \subseteq Q\}$.

If $m = n = 1$, then the statement is obvious. Assume $m + n > 2$, or equivalently, $|\max(H)| > 0$, and let $H \in \max(I_R)$. We consider three cases.

Case 1. $|\max(H)| = 1$. Then $\max(H) = \{R_1\}$ and R_1 is $(m - 1) \times n$ or $m \times (n - 1)$. By induction it follows $|H| = 1 + \left|H|_{R_1}\right| \geq 1 + ((m + n - 1) - 1) = m + n - 1$.

Case 2. $|\max(H)| = 2$. Then $\max(H) = \{R_1, R_2\}$, and w.l.o.g. we can assume that R_1 is $m_1 \times n$ and R_2 is $m_2 \times n$ where $m = m_1 + m_2 + 1$. By induction, we can write $|H| = 1 + \left|H|_{R_1}\right| + \left|H|_{R_2}\right| \geq 1 + (m_1 + n - 1) + (m_2 + n - 1) = m + n - 1 + (n - 1) \geq m + n - 1$. Note that the last inequality is strict unless $n = 1$.

Case 3. $|\max(H)| \geq 3$. Let $\{R_1, R_2, \dots, R_k\}$ be the set of maximal elements of $H - \{R\}$ that are at the border of R, and let R_i be $u_i \times v_i$ for $i = 1, \dots, k$. Further, let $w_1, w_2, \dots, w_{k+4}$ denote the lengths of the sides of the R_i that fall on the border of R (see Fig. 1). It is easy to see that, since$H \in \max(I_R)$, the gap between neighboring members of $\{R_1, R_2, \dots, R_k\}$ is exactly 1. Thus, we have

$$k + \sum_{j=1}^{k+4} w_j = 2m + 2n.$$

Since the w_j account for both dimensions of a rectangle R_i only when R_i is at a corner of R, we also have

$$\sum_{i=1}^{k} (u_i + v_i) \geq (k - 4) + \sum_{j=1}^{k+4} w_j,$$

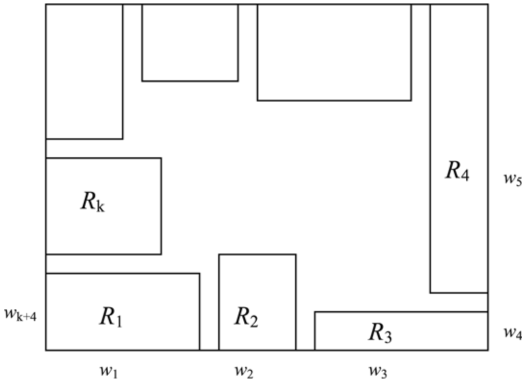

Fig. 1. Rectangles at the border of R.

which together with the last equation yields

$$\sum_{i=1}^{k}(u_i + v_i) \geq 2m + 2n - 4.$$

If $k > m + n - 2$, then obviously, $|H| \geq k + 1 > m + n - 1$. Now assuming $k \leq m + n - 2$, and applying induction, we can write

$$\begin{aligned}|H| &\geq 1 + \sum_{i=1}^{k} \left|H|_{R_i}\right| \geq 1 + \sum_{i=1}^{k}(u_i + v_i - 1) = 1 + \sum_{i=1}^{k}(u_i + v_i) - k \\ &\geq 1 + 2m + 2n - 4 - k = (m + n - 1) + (m + n - k - 2) \geq m + n - 1.\end{aligned}$$

Note that in this case also, the inequality is strict. For equality to hold, we need $m+n-k-2 = 0$. In addition, using the fact that no R_i can be 1×1, we would get

$$2m + 2n = k + \sum_{j=1}^{k+4} w_j \geq k + (k + 4) + 4 = 2m + 2n + 4,$$

a contradiction. □

Proposition 2. *For $H \in \max(I_R)$, we have $|H| = g(m, n) = m + n - 1$ only when H is a sequence of rectangles each contained in the next except possibly for the first $m - 1$ (or $n - 1$), all contained in the mth (or nth) rectangle which is $m \times 1$ (or $1 \times n$).*

Proof. This can be seen by examining the proof of Proposition 1. □

3. Values between the minimum and the maximum

Proposition 3. *For any integer h with $g(m, n) \leq h \leq f(m, n)$, there is a system of rectangular islands $H \in \max(I_R)$ such that $|H| = h$.*

Proof. Proceed by induction on $m + n$ by noting first that the few cases when $m + n \leq 6$ can all easily be checked. Thus, assume $m + n \geq 7$. Then at least one of m and n, say n, is greater than or equal to 4. Let H_0 and H_1 be two members of $\max(I_R)$ with the following partial definitions:

$$\max(H_0) = \{R_0\} \quad \text{where } R_0 = \{(x, y) \in R : y \leq n - 1\},$$

and

$$\max(H_1) = \{R_1, R_2\} \quad \text{where}$$
$$R_1 = \{(x, y) \in R : y \leq n-2\} \text{ and } R_2 = \{(x, y) \in R : n-1 \leq y \leq n\}.$$

By the inductive hypothesis, for any given h_0 with $g(m, n-1) \leq h_0 \leq f(m, n-1)$, we can define $H_0|_{R_0} \in \max(I_{R_0})$ so that $\left|H_0|_{R_0}\right| = h_0$. Similarly, for any given h_1 with $g(m, n-2) \leq h_1 \leq f(m, n-2)$, we can define $H_1|_{R_1} \in \max(I_{R_1})$ so that $\left|H_1|_{R_1}\right| = h_1$. Then we obtain

$$g(m, n) = g(m, n-1) + 1 \leq |H_0| = h_0 + 1 \leq f(m, n-1) + 1,$$

and

$$g(m, n-2) + m + 1 \leq |H_1| = h_1 + m + 1 \leq f(m, n-2) + m + 1 = f(m, n).$$

Thus, as h_0 runs through the integers in the interval $[g(m, n-1), f(m, n-1)]$, the value of $|H_0|$ ranges through the integers in the interval $[g(m, n), f(m, n-1)+1]$, and similarly, as h_1 runs through the integers in the interval $[g(m, n-2), f(m, n-2)]$, the value of $|H_1|$ ranges through the integers in $[g(m, n-2)+m+1, f(m, n)]$. Hence, we will obtain any integer $h \in [g(m, n), f(m, n)]$ as the size of an appropriately defined H_0 or H_1 if $f(m, n-1)+1 \geq g(m, n-2)+m$, or $f(m, n-1)+1-g(m, n-2)-m \geq 0$. The latter will follow if using

$$f(m, n-1) = \left\lfloor \frac{m(n-1)+m+(n-1)-1}{2} \right\rfloor = \left\lfloor \frac{mn+n-2}{2} \right\rfloor \geq \frac{mn+n-3}{2}$$

we can show that $\frac{mn+n-3}{2} + 1 - g(m, n-2) - m \geq 0$. However,

$$\begin{aligned} \frac{mn+n-3}{2} + 1 - g(m, n-2) - m &= \frac{mn+n-3}{2} + 1 - (m+n-2-1) - m \\ &= \frac{(m-1)(n-4)+1}{2} \geq 0, \end{aligned}$$

since we have $n \geq 4$. □

4. Extensions

There are at least two possible extensions that one might wish to explore both in the case of $g(m, n)$ and in the case of $f(m, n)$ as well. One, also mentioned in Czédli [1], is considering the analogous systems in higher dimensions. Another is obtained by requiring a gap of size at least k, instead of 1, between two neighboring rectangles.

References

[1] G. Czédli, The number of rectangular islands by means of distributive lattices, European J. Combin. 30 (1) (2009) 208–215.